Singapore

IMO

National Team Selection Tests

1994–1998

1994/95

1.1.* Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of all natural numbers and $f : \mathbb{N} \to \mathbb{N}$ be a function. Suppose $f(1) = 1$, $f(2n) = f(n)$ and $f(2n+1) = f(2n) + 1$ for all natural numbers n.

(i) Calculate the maximum value M of $f(n)$ for $n \in \mathbb{N}$ with $1 \leq n \leq 1994$.

(ii) Find all $n \in \mathbb{N}$, with $1 \leq n \leq 1994$, such that $f(n) = M$.

1.2. ABC is a triangle with $\angle A > 90^o$. On the side BC, two distinct points P and Q are chosen such that $\angle BAP = \angle PAQ$ and $BP \cdot CQ = BC \cdot PQ$. Calculate the size of $\angle PAC$.

1.3. In a dance, a group S of 1994 students stand in a big circle. Each student claps the hands of each of his two neighbours a number of times. For each student x, let $f(x)$ be the total number of times x claps the hands of his neighbours. As an example, suppose there are 3 students A, B and C. A claps hand with B two times, B claps hand with C three times and C claps hand with A five times. Then $f(A) = 7, f(B) = 5$ and $f(C) = 8$.

(i) Prove that $\{f(x) \mid x \in S\} \neq \{n \mid n \text{ is an integer}, 2 \leq n \leq 1995\}$.

(ii) Find an example in which

$$\{f(x) \mid x \in S\} = \{n \mid n \text{ is an integer}, n \neq 3, 2 \leq n \leq 1996\}.$$

2.1. Let $f(x) = \frac{1}{1+x}$ where x is a positive real number, and for any positive integer n, let

$$g_n(x) = x + f(x) + f(f(x)) + \cdots + f(f(\ldots f(x))),$$

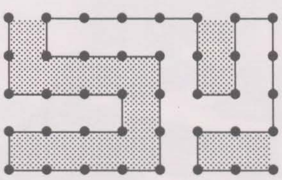
the last term being f composed with itself n times. Prove that

(i) $g_n(x) > g_n(y)$ if $x > y > 0$.

(ii) $g_n(1) = \frac{F_1}{F_2} + \frac{F_2}{F_3} + \cdots + \frac{F_{n+1}}{F_{n+2}}$, where $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

2.2. Let ABC be an acute-angled triangle. Suppose that the altitude of $\triangle ABC$ at B intersects the circle with diameter AC at P and Q, and the altitude at C intersects the circle with diameter AB at M and N. Prove that P, Q, M and N lie on a circle.

2.3. Show that a path on a rectangular grid which starts at the northwest corner, goes through each point on the grid exactly once, and ends at the southeast corner divides the grid into two equal halves: (a) those regions opening north or east; and (b) those regions opening south or west.



(The figure above shows a path meeting the conditions of the problem on a 5×8 grid. The shaded regions are those opening north or east while the rest open south or west.)

1995/96

1.1. Let P be a point on the side AB of a square $ABCD$ and Q a point on the side BC. Let H be the foot of the perpendicular from B to PC. Suppose that $BP = BQ$. Prove that QH is perpendicular to HD.

1.2. For each positive integer k, prove that there is a perfect square of the form $n2^k - 7$, where n is a positive integer.

1.3. Let $S = \{0, 1, 2, \ldots, 1994\}$. Let a and b be two positive numbers in S which are relatively prime. Prove that the elements of S can be arranged into a sequence $s_1, s_2, s_3, \ldots, s_{1995}$ such that $s_{i+1} - s_i \equiv \pm a$ or $\pm\, b \pmod{1995}$ for $i = 1, 2, \ldots, 1994$.

2.1. Let C, B, E be three points on a straight line l in that order. Suppose that A and D are two points on the same side of l such that

(i) $\angle ACE = \angle CDE = 90^o$ and

(ii) CA = CB = CD.

Let F be the point of intersection of the segment AB and the circumcircle of $\triangle ADC$. Prove that F is the incentre of $\triangle CDE$.

2.2. Prove that there is a function f from the set of all natural numbers to itself such that for any natural number n, $f(f(n)) = n^2$.

2.3. Let S be a sequence $n_1, n_2, \ldots, n_{1995}$ of positive integers such that $n_1 + \cdots + n_{1995} = m < 3990$. Prove that for each integer q with $1 \leq q \leq m$, there is a sequence $n_{i_1}, n_{i_2}, \ldots, n_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq 1995$, $n_{i_1} + \cdots + n_{i_k} = q$ and k depends on q.

1996/97

1.1. Let ABC be a triangle and let D, E and F be the midpoints of the sides AB, BC and CA respectively. Suppose that the angle bisector of $\angle BDC$ meets BC at the point M and the angle bisector of $\angle ADC$ meets AC at the point N. Let MN and CD intersect at O and let the line EO meet AC at P and the line FO meet BC at Q. Prove that $CD = PQ$.

1.2. Let a_n be the number of n-digit integers formed by 1, 2 and 3 which do not contain any consecutive 1's. Prove that a_n is equal to $(\frac{1}{2} + \frac{1}{\sqrt{3}})(\sqrt{3} + 1)^n$ rounded off to the nearest integer.

1.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function from the set $\mathbb{R}$ of real numbers to itself. Find all such functions f satisfying the two properties:

(a) $f(x + f(y)) = y + f(x)$ for all $x, y \in \mathbb{R}$,

(b) the set $\left\{ \dfrac{f(x)}{x} \ : \ x \text{ is a nonzero real number} \right\}$ is finite.

2.1. Four integers a_0, b_0, c_0, d_0 are written on a circle in the clockwise direction. In the first step, we replace a_0, b_0, c_0, d_0 by a_1, b_1, c_1, d_1, where $a_1 = a_0 - b_0, b_1 = b_0 - c_0, c_1 = c_0 - d_0, d_1 = d_0 - a_0$. In the second step, we replace a_1, b_1, c_1, d_1 by a_2, b_2, c_2, d_2, where $a_2 = a_1 - b_1, b_2 = b_1 - c_1, c_2 = c_1 - d_1, d_2 = d_1 - a_1$. In general, at the kth step, we have numbers a_k, b_k, c_k, d_k on the circle where $a_k = a_{k-1} - b_{k-1}, b_k = b_{k-1} - c_{k-1}, c_k = c_{k-1} - d_{k-1}, d_k = d_{k-1} - a_{k-1}$. After 1997 such replacements, we set $a = a_{1997}, b = b_{1997}, c = c_{1997}, d = d_{1997}$. Is it possible that all the numbers $|bc - ad|, |ac - bd|, |ab - cd|$ are primes ? Justify your answer.

2.2. For any positive integer n, evaluate

$$\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-i+1}{i},$$

where $\binom{m}{k} = \frac{m!}{k!(m-k)!}$ and $\lfloor \frac{n+1}{2} \rfloor$ is the greatest integer less than or equal to $\frac{n+1}{2}$.

2.3. Suppose the numbers $a_0, a_1, a_2, \ldots, a_n$ satisfy the following conditions:

$$a_0 = \frac{1}{2}, \quad a_{k+1} = a_k + \frac{1}{n} a_k^2 \quad \text{for} \quad k = 0, 1, \ldots, n-1.$$

Prove that $1 - \frac{1}{n} < a_n < 1$.

1997/98

1.1. Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$ and $EF = FA$. Prove that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}.$$

When does the equality occur?

1.2. Let $n \geq 2$ be an integer. Let S be a set of n elements and let A_i, $1 \leq i \leq m$, be distinct subsets of S of size at least 2 such that

$$A_i \cap A_j \neq \emptyset,\ A_i \cap A_k \neq \emptyset,\ A_j \cap A_k \neq \emptyset \quad \text{imply} \quad A_i \cap A_j \cap A_k \neq \emptyset.$$

Show that $m \leq 2^{n-1} - 1$.

1.3. Suppose $f(x)$ is a polynomial with integer coefficients satisfying the condition

$$0 \leq f(c) \leq 1997 \quad \text{for each } c \in \{0, 1, \ldots, 1998\}.$$

Is is true that $f(0) = f(1) = \cdots = f(1998)$?

2.1. Let I be the centre of the inscribed circle of the non-isosceles triangle ABC, and let the circle touch the sides BC, CA, AB at the points A_1, B_1, C_1 respectively. Prove that the centres of the circumcircles of $\triangle AIA_1$, $\triangle BIB_1$ and $\triangle CIC_1$ are collinear.

2.2. Let $a_1 \geq \cdots \geq a_n \geq a_{n+1} = 0$ be a sequence of real numbers. Prove that

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n} \sqrt{k}\left(\sqrt{a_k} - \sqrt{a_{k+1}}\right).$$

2.3. Let p and q be distinct positive integers. Suppose p^2 and q^3 are terms of an infinite arithmetic progression whose terms are positive integers. Show that the arithmetic progression contains the sixth power of some integer.

(**The numbering 1.1 refers to the first question of the selection test in the first day, while 2.1 refers to the first question of the selection test in the second day.*)